Symplectic geometry in Frobenius manifolds and quantum cohomology

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Abstract

The multiplication of vector fields on a Frobenius manifold M defines a Lagrangian submanifold of $T^{*}M$. In this paper, we give a proof of this "folklore" fact based on the formalism of Higgs pairs and we explain how it can be applied in the quantum cohomology situation (following Givental and Kim (1995)). © 1998 Elsevier Science B.V.

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0. Introduction

The notion of a Frobenius manifold has been the focus of extensive work and papers of Dubrovin in the recent years. This notion "transcends an enormous number of mathematical boundaries". In addition to the panoramic paper [5], some geometrical aspects are described in [12] (from which paper comes the above quotation), further relations with isomonodromy deformations and period mappings being developed in [18].

It is worthwhile mentioning that, according to Givental [7], the mirror symmetry conjecture can be understood as a correspondence between two types of Frobenius manifolds (topological σ-models, or quantum cohomology, vs. Landau–Ginsburg models, or deformations of singularities).

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Recall that a Frobenius structure on a manifold is an intricate mixture of various structures, mainly;
- a flat metric,
- a ring structure on the sheaf of vector fields,
- a vector field,
subject to a couple of compatibility conditions.

Some of the geometrical features of the Frobenius manifold are nicely described using flat coordinates determined by the metric, for instance, the fact that Frobenius manifolds provide solutions of the WDVV equations. For some others, it is more convenient to use a different set of coordinates, the so-called "canonical coordinates", which are mostly defined by the product.

The aim of this paper is modest: in some sense, it is only devoted to these "canonical coordinates". More precisely, I will try to extract, from all the geometric features present on a Frobenius manifold, those which do not depend on the metric. Also, I will try to avoid working in coordinates and will work from the global point of view.

In this spirit, the existence of canonical coordinates can be expressed as "the multiplication of vector fields on a Frobenius manifold M defines a Lagrangian subvariety of the cotangent bundle T^*M".

In Section 1, we will give a proof of the Lagrangian property based on the formalism of Higgs pairs (see for example [5,10,12] for alternative proofs).

I will then explain (following Givental and Kim [10]) how to apply this property to the quantum cohomology situation (by the way, I will also explain what the Frobenius structure on quantum cohomology is).

This paper is expository. It should be considered as an advertisement for the beautiful ideas contained in the papers mentioned above.

In this paper, the manifold M will be assumed to be complex algebraic (in applications, M will be affine, more precisely a vector space or a complex torus). All vector bundles will be complex.

1. Higgs pairs, spectral covers and Frobenius structures

1.1. Generalities on Higgs pairs and spectral covers

In general (see [22]), a *Higgs pair* over a manifold M is a pair (E, Ω), where $E \to M$ is a vector bundle and $\Omega : TM \to \text{End}(E)$ is a morphism such that $\Omega \wedge \Omega = 0$. This notion goes back to Hitchin's systems [11], where M was a curve and (E, Ω) was considered as an element of the cotangent bundle $T^*\mathcal{N}$ of the moduli space of stable bundles over M. Notice that the condition $\Omega \wedge \Omega = 0$ means that the various endomorphisms $\Omega_\xi(\alpha)$ (for $\alpha \in T_\xi M$) commute, a condition which is automatically fulfilled if $\dim M = 1$.

Any Higgs pair defines, via the eigenvalues of Ω, a *spectral cover* (see [4]), i.e., a subvariety $L \subset T^*M$ such that the projection

$$L \subset T^*M \xrightarrow{\pi} M$$

is a finite map (its degree is the rank N of the vector bundle E). Here is the (algebroge-ometrical) definition. As $\Omega \wedge \Omega = 0$, the morphism $\Omega : TM \rightarrow \text{End}(E)$ extends to a morphism

$$\text{Sym}(TM) \longrightarrow \text{End}(E)$$

by

$$(\xi, \alpha_1 \bullet \cdots \bullet \alpha_m) \cdot \beta = (\Omega_\xi(\alpha_1) \circ \cdots \circ \Omega_\xi(\alpha_m))(\beta).$$

Now, the sheaf of local sections of $\text{Sym}(TM)$ is nothing other than the sheaf $\pi_* \mathcal{O}_{T^\star M}$ of regular functions on the cotangent bundle $T^\star M$: a local section $\alpha_1 \bullet \cdots \bullet \alpha_m$ of $\text{Sym}(TM)$ defines a function on $T^\star M$ by

$$\alpha_1 \bullet \cdots \bullet \alpha_m(\xi, \mu) = \langle \mu, \alpha_1(\xi) \rangle \cdots \langle \mu, \alpha_m(\xi) \rangle$$

(ξ is a point of M, μ a linear form on $T_\xi M$ and the $\alpha_i(\xi)$'s are vectors in $T_\xi M$). Thus the $\mathcal{O}_M$-module E gets an $\mathcal{O}_{T^\star M}$-module structure. Our spectral cover L is just the support of E as an $\mathcal{O}_{T^\star M}$-module.

Set-theoretically, this is to say that a point (ξ, μ) of $T^\star M$ is in L if and only if, for any (local) function $f : T^\star M \rightarrow \mathbb{C}$ that annihilates local sections of E, $f(\xi, \mu) = 0$.

Let α be a vector field defined on an open subset U of M. Associated with α is a section

$$P_\alpha : (\xi, \mu) \longmapsto \det(\Omega_\xi(\alpha) - \mu_\xi(\alpha) \,\text{Id}_\xi)$$

of $\pi^\star \Lambda^N E$ over $T^\star U \subset T^\star M$. Assuming that there is at least some local vector field α for which the minimal and characteristic polynomials coincide, the spectral cover can also be defined by the vanishing of all the P_α's.

Assume that at some point $\xi_0 \in M$, all the $\Omega_\xi(\alpha)$ for $\alpha \in T_\xi M$ are diagonalisable. Over a neighbourhood of ξ_0, the bundle $E \rightarrow M$ then splits as a sum

$$E = \bigoplus_{i=1}^N W_i \tag{1}$$

of line bundles that are the eigenline bundles of Ω (recall that the $\Omega_\xi(\alpha)$ commute so that they can be simultaneously diagonalised). To each W_i and each $\alpha \in T_\xi M$ corresponds an eigenvalue $\mu_i(\alpha, \xi)$ of $\Omega_\xi(\alpha)$. Such a μ_i can be considered as a local section of $T^\star M$ and altogether, the μ_i's define a subvariety L with a degree N map to M.

Remark. The spectral cover (and the projection onto M) has singularities over the complement of the set of semi-simple points.

Let us call such a point ξ_0 a *semi-simple point*. If moreover, for some α, the eigenvalues of $\Omega_{\xi_0}(\alpha)$ are all distinct, we will call it a *regular* semi-simple point.

1.2. Frobenius bundles and spectral Lagrangians

Assume we are given a local trivialisation of E, so that Ω is indeed a 1-form *matrix*. Assume that Ω is closed ($d\Omega = 0$).

Lemma 1.1. *If $d\Omega = 0$, near a regular semi-simple point, L is a Lagrangian subvariety of T^*M.*

Proof. Let ξ_0 be a semi-simple point. On a suitable neighbourhood of ξ_0, choose a generator w_i for each line bundle W_i in the decomposition (1). Let μ_i be the corresponding eigenvalue, viewed as a local section of T^*M. The image of μ_i is one of the N branches of L over the neighbourhood of ξ_0 we are considering. To say that L is Lagrangian is to say that all μ_i's are closed 1-forms on M. We can now differentiate the relation

$$\Omega \cdot w_j = \mu_j w_j$$

to get

$$(d\Omega) \cdot w_j + \Omega \cdot dw_j = (d\mu_j)w_j + \mu_j \, dw_j. \tag{2}$$

Our assumption is that $d\Omega = 0$. Let us look at dw_j. It can be written in the same basis as

$$dw_j = \sum_{i=1}^{N} a_j^i w_i$$

for some 1-forms a_j^i, so that

$$\Omega \cdot dw_j - \mu_j \, dw_j = \sum_{i=1}^{N} a_j^i (\mu_i - \mu_j)w_i.$$

Relation (2) then tells us that

$$\sum_{i=1}^{N} a_j^i (\mu_i - \mu_j)w_i = (d\mu_j)w_j.$$

Assume now that the semi-simple point ξ_0 is regular, so that the μ_i's are distinct. Equating coefficients gives $a_j^i = 0$ for $i \neq j$ and $d\mu_j = 0$, which is precisely what we wanted to prove.   □

Remark. The proof given in [10] (due to Reshetikin) shows that the branches of the spectral cover associated with simple eigenvalues are Lagrangian. This is a little bit more general than what we have here.

Now, to globalise this property, the only thing we need to do is to give an intrinsic meaning to the closedness of Ω. This is achieved by the choice of a flat connection (if this exists)

on E, as this gives a prefered class of local trivialisations of the bundle and an exterior derivative d_∇.

Definition 1.2. A triple (E, Ω, ∇) is a Frobenius bundle if (E, Ω) is a Higgs pair, ∇ is a flat connection on E and $d_\nabla \Omega = 0$.

The global version of Lemma 1.1 is then:

Proposition 1.3. *Let (E, Ω, ∇) be a Frobenius bundle over M. If there exists a regular semi-simple point in M, the spectral cover is a Lagrangian subvariety of T^*M.*

Proof. If there is one regular semi-simple point, almost all points are regular semi-simple. Moreover, it suffices to prove the property over a neighbourhood of such a point.   □

Remark. Note that, by its very definition, the canonical (Liouville) form λ on T^*M, restricted to the local branch of L defined by w, is just the eigenvalue μ. In other words, if we call f_w the inclusion of the local branch of L in T^*M,

$$\Omega \cdot w = (f_w^* \lambda) w.$$

Restriction to a subvariety and symplectic reduction. Suppose now that $j : B \rightarrow M$ is the inclusion of a submanifold, so that $j^* T^*M$ is a co-isotropic subvariety of T^*M. At least at semi-simple points, the spectral cover is locally the graph of a 1-form and thus is transversal to $j^* T^*M$.

This is a simple form of the symplectic reduction process that the intersection $L \cap (j^* T^*M)$ projects to a *Lagrangian* subvariety $L_B \subset T^*B$. Of course, we have:

Proposition 1.4. *The Lagrangian subvariety L_B is the spectral Lagrangian associated with the Frobenius bundle $(j^*E, j^*\Omega, j^*\nabla)$ over B.*

Proof. The spectral cover for

$$j^*\Omega : B \longrightarrow \mathrm{End}(j^*E)$$

is defined by the eigenvalues $\mu_i(\alpha, \xi)$ of $\Omega_\xi(\alpha)$ for $\xi \in B$, $\alpha \in T_\xi B \subset T_\xi M$. In other words, one considers the section μ_i of T^*M as a section of T^*B by restriction and this is the definition of the symplectic reduction explained above.   □

1.3. The Higgs pair associated with a ring structure on vector fields

Suppose now that M is a manifold such that each tangent space $T_\xi M$ is endowed with a ring structure $\star_\xi$, such that:
1. The formula

$$\Omega_\xi(\alpha) \cdot \beta = \alpha \star_\xi \beta$$

(in which $\xi \in M$, $\alpha \in T_\xi M$, so that $\Omega_\xi(\alpha)$ is an endomorphism of $T_\xi M$ and $\Omega_\xi(\alpha) \cdot \beta$ is the image of the element $\beta \in T_\xi M$) defines a 1-form Ω with values in End(TM), i.e., a morphism

$$\Omega : TM \longrightarrow \text{End}(TM).$$

2. All the identities 1_ξ of the various tangent spaces fit together into a (global) vector field 1. Associativity and commutativity of $\star_\xi$ imply:

Lemma 1.5. $\Omega \wedge \Omega = 0$.

Proof. By definition, $\Omega_\xi(\alpha) \cdot (\Omega_\xi(\beta) \cdot \gamma) = \alpha \star_\xi (\beta \star_\xi \gamma)$, so that

$$(\Omega \wedge \Omega)_\xi(\alpha, \beta) \cdot \gamma = \tfrac{1}{2}[\Omega_\xi(\alpha), \Omega_\xi(\beta)] \cdot \gamma$$
$$= \tfrac{1}{2}[\alpha \star_\xi (\beta \star_\xi \gamma) - \beta \star_\xi (\alpha \star_\xi \gamma)]. \qquad \square$$

The products $\star_\xi$ give the sheaf Θ_M of local sections of TM (vector fields) a ring structure. We know that (TM, Ω) is a Higgs pair over M, let us now look at the associated spectral cover $L \subset T^*M$. As above, it is defined by

$$\Omega : \text{Sym}(TM) \longrightarrow \text{End}(TM),$$

but the identity element of the ring structure of $T_\xi M$ gives us a map

$$\text{End}(TM) \longrightarrow TM$$
$$\varphi \longmapsto \varphi(1)$$

such that the composition

$$\text{Sym}(TM) \longrightarrow \text{End}(TM) \longrightarrow TM$$

is just the product

$$(\xi, \alpha_1 \bullet \cdots \bullet \alpha_m) \longmapsto (\xi, \alpha_1 \star_\xi \cdots \star_\xi \alpha_m).$$

Thus Θ_M can be considered as the ring sheaf of regular functions on some subvariety of T^*M. Notice that the morphism $\text{Sym}(TM) \to TM$ is obviously onto (as a sheaf morphism) so that the inclusion $L \hookrightarrow T^*M$ is a closed immersion in the algebro-geometrical sense.

Proposition 1.6. *The ring sheaf $(\Theta_M, \star)$ is the sheaf of regular functions on the spectral cover L associated with (TM, Ω).*

Proof. This is almost tautological. As L is defined as the support of Θ_M as an $\mathcal{O}_{T^*M}$-module, a function $f : T^*M \to \mathbb{C}$ vanishes on L if and only if it annihilates Θ_M. But this means that, changing $\bullet$'s to $\star$'s inside the polynomial f,

$$f \star \alpha = 0 \quad \forall \alpha \in \Theta_M.$$

It remains to use the identities 1_ξ of our rings $T_\xi M$ to get the result. $\qquad \square$

The Lagrange property. So far, we have only used the ring structure on $T_\xi M$ and more precisely, to summarise:
- associativity and commutativity of $\star_\xi$ imply that (TM, Ω) is a Higgs pair (Lemma 1.5).
- identity 1_ξ to interpret functions on $T^\star M$ as vector fields on M and the spectral cover L as the spectrum of the ring $(\Theta_M, \star)$.

Let us now assume that M is endowed with a flat connection ∇ such that $d_\nabla \Omega = 0$. This is the case for instance when M is a Frobenius manifold (see [5,12] and below).

Proposition 1.7. *Assume there is a point $\xi_0 \in M$ such that the ring $T_{\xi_0}M$ is a semi-simple ring. If $\nabla\Omega = 0$, then the spectral cover is Lagrangian.*

Remark.
- The assumption means that, for any ξ in a neighbourhood of ξ_0, there exists a basis $(w_1(\xi), \ldots, w_N(\xi))$ of $T_\xi M$ such that

$$w_i(\xi) \star_\xi w_j(\xi) = \delta_{i,j} w_i(\xi)$$

for some non-zero $\mu_i(\xi)$: the algebra $T_\xi M$ splits as a sum $\bigoplus_{i=1}^N \mathbb{C} \cdot w_i$.
- Notice that this semi-simplicity property would be a consequence of the fact that there exists an element α in $T_{\xi_0}M$ such that the endomorphism $\Omega_{\xi_0}(\alpha) = \alpha \star_{\xi_0} \cdot$ has distinct eigenvalues.
- Although our spectral cover usually has singularities, [2] the Lagrange property needs only be checked at non-singular points.

Proof of Proposition 1.7. This would follow from Propositions 1.3 and 1.6 if ξ_0 were a regular semi-simple point. [3] In the case at hand, where Ω is defined by a semi-simple ring structure, one can replace the argument in the proof of Proposition 1.3 by the following one. Differentiate $w_j \star w_j = w_j$ to get

$$2\,dw_j \star w_j = dw_j.$$

Use then that $w_i \star w_j = 0$ for $i \neq j$ to get

$$2a_j^i w_j = \sum_{i=1}^N a_j^i w_i,$$

so that $dw_j = 0$, and (2) with $d\Omega = 0$ gives

$$(d\mu_j)w_j = 0. \qquad \square$$

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[2] However, accoding to Proposition 1.6, the structural sheaf is the sheaf of sections of a vector bundle, so that the singularities are rather soft.

[3] As pointed out to me by E. Markman, in this case semi-simplicity implies regular semi-simplicity.

1.4. Frobenius manifolds

If we mix the structures described in Sections 1.2 and 1.3 together, we eventually get the *Frobenius manifold* structure (see [5,12]). Here is the definition we shall use:

Definition 1.8. A pre-Frobenius manifold structure on the manifold M is a Frobenius structure (TM, Ω, ∇) on its tangent bundle such that the formula

$$\alpha \star_\xi \beta = \Omega_\xi(\alpha) \cdot \beta$$

defines unital ring structures on the tangent spaces $T_\xi M$.

Remark. Notice that associativity, existence of identity and $\Omega \wedge \Omega = 0$ imply commutativity.

Frobenius specialists would insist on various other structures, e.g. ∇ should be the Levi-Civita connection associated with a flat metric on M and there should be an *Euler vector field* η. This is a vector field which rescales all the structures. [4] It satisfies, in particular

$$\mathcal{L}_\eta \Omega = \Omega$$

and this implies homogeneity of the Lagrangian subvariety with respect to η. Also, in the Frobenius world language, a manifold with a Frobenius structure satisfying the semi-simplicity assumption above is called *massive*. In this language, Proposition 1.7 implies:

Corollary 1.9. *The spectral cover associated with a massive Frobenius manifold is a (homogeneous) Lagrangian subvariety.*

Canonical coordinates. At a semi-simple point, the eigenvalues μ_i define N independent closed 1-forms on the N-manifold M. Local primitives $(x_1, \ldots x_N)$ give coordinates on M which are what Frobenius people use to call *canonical coordinates*, so that Corollary 1.9 is just a global way to state the existence of these coordinates. In canononical coordinates
– our w_i's are the $\partial/\partial x_i$ and our identity vector field is

$$1_\xi = \sum_{i=1}^N w_i(\xi) = \sum_{i=1}^N \frac{\partial}{\partial x_i}$$

(note that the w_i's commute in the Lie algebra of vector fields because the μ_i's are closed 1-forms),
– our matrix Ω is just the diagonal $(dx_1, \ldots, dx_N)$,
– moreover, for any choice of the primitives $(x_1, \ldots, x_N)$, the (local) vector field $\eta = \sum x_i \partial/\partial x_i$ satisfies $\mathcal{L}_\eta \Omega = \Omega$.

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[4] The Euler vector field η and the identity vector field 1 generate an infinitesimal action of the affine group of the line on the Frobenius manifold.

Also, Ω being closed implies that it can be written locally as the derivative of some mapping

$$S : M \longrightarrow \text{End}(TM).$$

This is where the potential of the Frobenius manifold comes from.

The metric and the potential. As we have already mentioned it, in the Frobenius landscape, there is also a *flat* metric on the manifold M, the flat connection used so far being its Levi-Civitá connection. It has to satisfy a couple of compatibility conditions among which the compatibility with the product, which can be put in the following form: for any $\xi \in M$, any $\alpha \in T_\xi M$, $\Omega_\xi(\alpha)$ is a self-adjoint endomorphism of $T_\xi M$.

With this assumption, working in flat coordinates, a primitive S of Ω can be assumed to be self-adjoint as well (replace S by $\frac{1}{2}(S + {}^tS)$ to get a self-adjoint primitive). Then, as a local section of the self-adjoint endomorphisms of TM, S must be the second derivative of some function $F : M \to \mathbb{C}$, which is the *potential* of the Frobenius structure.

The associativity and commutativity of the products $\star_\xi$ can be translated in a system of third-order partial differential equations, the *WDVV equation*.

1.5. A simple example: unfolding z^{n+1}

Baby–Hurwitz space. Before giving applications of these constructions to quantum cohomology, it is useful to look at a simple example. Let $\mathcal{E}$ be the affine space of all complex monic degree-$(n + 1)$ polynomials of the form

$$P(z) = z^{n+1} + a_n z^{n-1} + \cdots + a_1.$$

This belongs to (at least) two families of Frobenius manifolds: it can be considered first as the space of the universal deformation of the singularity z^{n+1} – and thus, it belongs to a family evocated in Section 1. Also, it is the space of all meromorphic functions on a genus-0 curve with a single order-$n + 1$ prescribed pole – and thus it is the simplest example of a Hurwitz space (a space of coverings of $\mathbb{P}^1$, see [5, Ch. V]).

The tangent space $T_P \mathcal{E}$ is canonically identified with the vector space E of polynomials of degree less than or equal to $n - 1$. To emphasise the dependence on P, we will prefer to write

$$T_P \mathcal{E} = \mathbb{C}[z]/\langle P'(z) \rangle$$

... this is of course a ring, so that we are indeed in the situation of Section 1.3.

The spectral cover. Eigenvalues of the multiplication are easily found: write that $\beta(z)$ is a (common) eigenvector of the multiplication $\star_P$:

$$\alpha(z) \star_P \beta(z) = \mu_P(\alpha)\beta(z),$$

i.e.,

$$\alpha(z)\beta(z) = \mu_P(\alpha)\beta(z) + P'(z)Q(z)$$

and evaluate at a root u of P' to get

$$\alpha(u)\beta(u) = \mu_P(\alpha)\beta(u)$$

so that the $\mu_P(\alpha) = \alpha(u)$ (for $P'(u) = 0$) are natural candidates for the eigenvalues. Computing more carefully gives:

Lemma 1.10. *If $P'(z) = (n + 1) \prod (z - u_i)^{m_i}$, the characteristic polynomial of the multiplication $\alpha \star_P \cdot$ is $(-1)^n \prod (z - \alpha(u_i))^{m_i}$.*

Proof. Let α be a degree $\leq n - 1$ polynomial. For any complex number u, decompose α according to the Taylor formula to get

$$\frac{\alpha(z)}{(z - u)^k} = \frac{\alpha(u)}{(z - u)^k} + \frac{\alpha'(u)}{(z - u)^{k+1}} + \cdots + \frac{1}{(k - 1)!} \frac{\alpha^{(k-1)}(u)}{(z - u)} + Q(z). \qquad (3)$$

If u is a multiplicity-m root of P', define m polynomials

$$\beta_k(z) = \frac{P'(z)}{(z - u)^k} \quad (1 \leq k \leq m).$$

Multiply both sides of (3) by $P'(z)$ to get

$$\alpha \star_P \beta_k = \alpha(u)\beta_k + \alpha'(u)\beta_{k-1} + \cdots + \frac{\alpha^{k-1}(u)}{(k - 1)!} \beta_1.$$

Thus the m-dimensional subspace of $T_P\mathcal{E}$ spanned by the β_k's is stable under multiplication $\star_P$ and the $\alpha\star_P$'s are (simultaneously) triangular in this basis, with $\alpha(u)$'s on the diagonal. $\qquad\qquad\square$

The spectral cover is the subvariety

$$L = \{(P, \mu) \in \mathcal{E} \times E^* \mid \exists u \in \mathbb{C} \text{ such that } P'(u) = 0 \text{ and } \mu(\alpha) = \alpha(u)\}.$$

Lagrange property. We do not need to find a connection and apply any previous result, as L is "obviously" Lagrangian, according to a neat generating function trick: let

$$S : \mathcal{E} \times \mathbb{C} \longrightarrow \mathbb{C}$$

be the evaluation mapping $(P, z) \mapsto P(z)$. Consider the subvariety of $\mathcal{E} \times \mathbb{C}$ defined by $d_z S = 0$:

$$\mathcal{L} = \{(P, z) \mid \mathcal{E} \times \mathbb{C} \mid P'(z) = 0\}$$

so that the image of

$$\mathcal{L} \longrightarrow T^*\mathcal{E}$$
$$(P, z) \longmapsto (P, d_P S)$$

is a Lagrangian subvariety of $T^\star\mathcal{E}$. As S is affine in the variable P,

$$(d_P S)_{P,z}(\alpha) = \alpha(z)$$

and this Lagrangian subvariety is our spectral cover.

Canonical coordinates. The generating function contains all desired information, e.g. local primitives of the eigenvalues – as these are written $d_P S$!

Proposition 1.11 (see e.g. [5,12]). *Near a polynomial P_0 with n distinct critical point $(u_1^0, \ldots, u_n^0)$, the n functions*

$$P \mapsto (P(u_1), \ldots, P(u_n))$$

are a set of canonical coordinates.

Here is the dual basis. For P in a suitable neighbourhood of P_0, let $(u_1, \ldots, u_n)$ be the n distinct roots of P'. According to the computation in Lemma 1.10,

$$v_i(z) = \frac{P'(z)}{z - u_i} \quad (1 \leq i \leq n)$$

constitute a basis of eigenvectors. By definition

$$v_i \star_P v_j = \delta_{i,j} v_j(u_j) v_j$$

and $v_j(u_j) = P''(u_j)$, so that the basis

$$w_i(z) = \frac{1}{P''(u_i)} \frac{P'(z)}{(z - u_i)} \quad (1 \leq i \leq n)$$

satisfies

$$w_i \star_P w_j = \begin{cases} w_j & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Other pieces of the Frobenius structure. First of all, there is a 1-form θ on $\mathcal{E}$, defined by

$$\theta_P(\alpha) = -\text{Res}_\infty\left(\frac{\alpha(z)\,dz}{P'(z)}\right).$$

It defines a non-degenerate bilinear form using the product:

$$\langle \alpha, \beta \rangle_P = \theta_P(\alpha \star_P \beta) = -\text{Res}_\infty\left(\frac{\alpha(z)\beta(z)\,dz}{P'(z)}\right).$$

At generic points, in the basis $(w_1, \ldots, w_n)$ constructed above,

$$\langle w_i, w_j \rangle_P = \begin{cases} \dfrac{1}{P''(u_j)} & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

so that the bilinear form is diagonal in canonical coordinates, where it has the expression

$$\sum_{j=1}^{n} \frac{\mathrm{d}x_j^2}{P''(u_j)},$$

as it should [12].

Notice that to produce all these data (in particular the cannoical coordinates) and to check their properties, we have used only basic properties of the polynomials and their roots. A more significant property of the metric above is that it is *flat*: there is another set of local coordinates in which the metric has constant coefficients.

In the second part of this paper, we will consider an example (quantum cohomology) in which the Frobenius manifold is an affine space and the natural linear coordinates are flat coordinates. This is not the case here, and it is not that obvious to construct the flat coordinates. The idea is to write

$$z(w, t) = w + \frac{t_1}{w} + \cdots + \frac{t_n}{w^n} + \cdots$$

and to solve

$$P(z(w, t)) = w^{n+1}$$

near $z = \infty$, thus defining local flat coordinates $(t_1, \ldots, t_n)$ (see [5,12,19]).

2. Applications to quantum cohomology

It is recent folklore (see [5,12,15]) that the quantum cohomology of a symplectic manifold should be a Frobenius manifold. Of course, this statement must be made more precise, as there are so many variants of "quantum cohomology". Moreover, this Frobenius structure is usually presented through the "Gromov–Witten potential", i.e., with computations in coordinates, as Frobenius people are interested in solutions of WDVV equations. We will try to precise the various quantum products being used and to present what we need of the Frobenius structure as geometrically as possible.

2.1. Gromov–Witten invariants

There are many variants of definitions of Gromov–Witten invariants in symplectic and/or algebraic geometry. In all cases, the formalism is rather heavy and I do not want to spend much time and place on it. I will use the most elementary approach – if not the more general. The aim here is just to fix the notation, which I will try to make coherent with that of [1]. I send the reader to [2,16] for the properties of holomorphic curves and to the original paper of Ruan and Tian [17] for precisions on Gromov–Witten invariants for weakly monotone manifolds and proofs.

More general constructions, working for all symplectic manifolds, rely on the beautiful idea of "stable maps" due to Kontsevich [13] and make use of a notion of fundamental class which can be rather sophisticated (see e.g. [6,14,20]).

Definitions. A symplectic manifold (W, ω) is *monotone* if its first Chern class c_1 is a positive multiple of the cohomology class of the symplectic form ω. It is *weakly monotone* if any homology class A such that $\langle \omega, A \rangle > 0$ and $\langle c_1, A \rangle \geq 3 - n$ satisfies $\langle c_1, A \rangle \geq 0$.

This property ensures that, for any generic almost complex structure J calibrated by ω, the first Chern class of X, evaluated on the class of a J-holomorphic sphere, is non-negative.

On a weakly monotone symplectic manifold (X, ω), Ruan and Tian define Gromov–Witten invariants as elements in bordism groups of pseudo-cycles $\Omega_*^{ps}(X^k \times X^l)$.

Given a class $A \in H_2(X; \mathbb{Z})$, denote by $\mathcal{M}^A(J, \nu)$ the space of solutions $u : \mathbb{P}^1 \to X$ of the modified Cauchy–Riemann equation $\overline{\partial}_J u = \nu$ which represent the class A. Denote by $z = (z_1, \ldots, z_k)$ a k-uple of distinct points in $\mathbb{P}^1$. Suppose now that $k \geq 3$ and $l \in \mathbb{N}$ are given, then $\Psi_{k,l}^A$ is defined as the class of the evaluation mapping

$$\mathrm{ev}_{z,l} : \mathcal{M}^A(J, \nu) \times \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{l \text{ times}} \longrightarrow \underbrace{X \times \cdots \times X}_{k \text{ times}} \times \underbrace{X \times \cdots \times X}_{l \text{ times}}$$

$$(u, \zeta_1, \ldots, \zeta_l) \longmapsto (u(z_1), \ldots, u(z_k), u(\zeta_1), \ldots, u(\zeta_l))$$

in $\Omega_{2n+2\langle c_1, A \rangle + 2l}^{ps}(X^k \times X^l)$ for a generic pair (J, ν) (see [17] or [1]).

If one forgets the assumption $k \geq 3$, one has to take into account the reparametrisation group G_k of $(\mathbb{P}^1, z)$ (which is trivial if $k \geq 3$), namely
- $PSL(2; \mathbb{C})$ if $k = 0$ (it has real dimension 6),
- the group of affine transformations of $\mathbb{C}$ if $k = 1$ (it has dimension 4),
- the group of dilations of $\mathbb{C}$ if $k = 2$ (it has dimension 2).

The group G_k has dimension $\sup(6 - 2k, 0)$, it acts on $\mathcal{M}^A(J, \nu) \times (\mathbb{P}^1)^l$ by

$$g \cdot (u, \zeta_1, \ldots, \zeta_l) = (u \circ g^{-1}, g\zeta_1, \ldots, g\zeta_l)$$

and the evaluation mapping factors through this action, defining

$$\mathrm{ev}_{z,l} : \mathcal{M}^A(J, \nu) \times_{G_k} (\mathbb{P}^1)^l \longrightarrow X^{k+l}.$$

Theorem 2.1 [16,17]. *Let (W, ω) be a weakly monotone symplectic manifold of dimension $2n$. Let A be an element of $H_2(X; \mathbb{Z})$. If $k \leq 2$, assume $A \neq 0$. Then for generic (J, ν), the evaluation mapping*

$$\mathrm{ev}_{z,l} : \mathcal{M}^A(J, \nu) \times_{G_k} (\mathbb{P}^1)^l \longrightarrow X^k \times X^l$$

defines a pseudo-cycle of dimension $2n + 2\langle c_1, A \rangle + 2l - \dim G_k$ and thus a morphism

$$\Psi_{k,l}^A : H_*(X^{k+l}) \longrightarrow \mathbb{Z}.$$

The intersection number

$$(\mathcal{M}^A(J, \nu) \times (\mathbb{P}^1)^l, \mathrm{ev}_{z,l}) \cdot (a_1 \otimes \cdots \otimes a_k \otimes b_1 \otimes \cdots \otimes b_l)$$

is denoted $\Psi_{k,l}^A(a_1 \otimes \cdots \otimes a_k | b_1 \otimes \cdots \otimes b_l)$. If $a \in H_*(X^k)$, $b \in H_*(X^l)$, we will also use the notation $\Psi_{k,l}^A(a | b)$ for $\Psi_{k,l}^A(a \otimes b)$.

We will need a few elementary properties of the Gromov–Witten invariants just defined.

Symmetry. Recall that, given a geometric holomorphic non-constant rational curve $C \subset X$, to fix the values $u(z_i)$ of a holomorphic parametrisation at three distinct points z_1, z_2 and z_3 of $\mathbb{P}^1$ actually determines the parametrisation. From this, one easily deduces:

Proposition 2.2. *Let A be a non-zero homology class. Assume $l \geq 1$. Then for any classes a, b, c and d in $H_\star(X)$, y in $H_\star(X^{l-1})$,*

$$\Psi^A_{3,l}(a \otimes b \otimes c | d \otimes y) = \Psi^A_{0,3+l}(|a \otimes b \otimes c \otimes d \otimes y).$$

Using the obvious symmetry properties of the Gromov–Witten invariants, one gets:

Corollary 2.3. *Let A be a non-zero homology class. Assume $l \geq 1$. Then for any classes a, b, c and d in $H_\star(X)$, $y \in H_\star(X^{l-1})$,*

$$\Psi^A_{3,l}(a \otimes b \otimes c | d \otimes y) = \Psi^A_{3,l}(a \otimes b \otimes d | c \otimes y).$$

Degree-2 classes. It is in the nature of the Gromov–Witten invariants that codimension-2 cycles play a special role. Here is a property that illustrates this assertion.

Proposition 2.4. *Let ξ be a degree-2 cohomology class. Let x be the Poincaré dual homology class. Then, for all $a \in H_\star(X^k)$, $b \in H_\star(X^l)$,*

$$\Psi^A_{k,l+1}(a | b \otimes x) = \langle \xi, A \rangle \Psi^A_{k,l}(a | b).$$

Proof. One wants to check that the diagram

$$
\begin{array}{ccc}
H_d(X^k \times X^l) & \longrightarrow & H_{d+2(n-1)}(X^k \times X^{l+1}) \\
\Psi^A_{k,l} \downarrow & & \downarrow \Psi^A_{k,l+1} \\
\mathbb{Z} & \xrightarrow{(x \cdot A)} & \mathbb{Z}
\end{array}
$$

(in which the top arrow is $y \mapsto y \otimes x$) is commutative, which is more or less obvious from the definition of the Gromov–Witten invariants:

$$(\mathcal{M}^A(J, \nu) \times (\mathbb{P}^1)^{l+1}, \mathrm{ev}_{z,l+1}) \cdot (y \otimes x) = ((\mathcal{M}^A(J, \nu), \mathrm{ev}_{z,l}) \cdot y)(u_\star[\mathbb{P}^1]).$$

2.2. Quantum cup-products

Notation. Assume as usual that $H_2(X; \mathbb{Z})$ is torsion free – or denote by $H_2(X; \mathbb{Z})$ the torsion free part of the second homology group. Let Λ be its group ring, $\Lambda = \mathbb{Z}[H_2(X; \mathbb{Z})]$, and denote q^A the (multiplicative) counterpart in Λ of $A \in H_2(X; \mathbb{Z})$.

If the symplectic manifold (X, ω) is only weakly monotone, replace Λ by an ad hoc completion Λ_ω, e.g. the Novikov ring associated with the symplectic form, viewed as a morphism

$$\omega : H_2(X; \mathbb{Z}) \longrightarrow \mathbb{R}.$$

We will use as systematically as possible the convention that Latin (resp. Greek) characters denote homology (resp. cohomology) classes, Poincaré duality exchanging corresponding characters (e.g. $D\alpha = a$, $D\xi = x$).

The quantum cup-product at a point ξ. The invariant $\Psi_{3,l}^{A}(a \otimes b \otimes c | x \otimes \cdots \otimes x)$ is non-zero only when

$$2n(l+3) - (\dim a + \dim b + \dim c + l \dim x) = 2n + 2\langle c_1, A \rangle + 2l,$$

and in particular, if $\operatorname{codim} x \neq 2$, for at most one value of l. If $\operatorname{codim} x = 2$, let ξ be the dual (degree-2) cohomology class and apply Proposition 2.4 to get

$$\sum_{l \geq 0} \frac{1}{l!} \Psi_{3,l}^{A}(a \otimes b \otimes c | x \otimes \cdots \otimes x) = \Psi_{3,0}^{A}(a \otimes b \otimes c) \exp\langle \xi, A \rangle.$$

In any case, the sum

$$\sum_{l \geq 0} \frac{1}{l!} \Psi_{3,l}^{A}(a \otimes b \otimes c | x \otimes \cdots \otimes x)$$

defines a complex number for all $A, a, b, c, x \in H_*(X)$. Eventually:

Proposition 2.5. *For $\alpha, \beta, \xi \in H^*(X)$, $c \in H_*(X)$, the equality*

$$\langle \alpha \star_{\xi,q} \beta, c \rangle = \sum_A \sum_l \frac{1}{l!} \Psi_{3,l}^{A}(a \otimes b \otimes c | x \otimes \cdots \otimes x) q^A$$

defines an element $\alpha \star_{\xi,q} \beta \in H^*(X) \otimes \Lambda_\omega$. Extending $\star_{\xi,q}$ by linearity gives $H^*(X) \otimes \Lambda_\omega$ the structure of a unital commutative (in the graded sense) ring.

The associativity is by no mean a triviality, but follows from the composition rules of [17]. The identity element is the identity element of the cup-product, i.e., the dual of the fundamental class.

There are many variants of definitions of "quantum cup-product". This is the most general version, as it depends on the point ξ, and, because of the variables q (and the use of the Novikov ring Λ_ω), there is no convergence problem in the definition.

There are basically two ways to get a less general structure: to specialise, either at a value of ξ, or at a value of q, the latter leading to the mentioned convergence problems.

- To specialise at $\xi = 0$ would give the "usual" quantum product, a formal series in q, denoted $\alpha \star \beta$, a ring structure on $QH^*(X) := H^*(X) \otimes \Lambda_\omega$.
- To specialise at $q = 1$, allowing ξ to be any cohomology class, but assuming the series defining $\alpha \star_{\xi,1} \beta$, that we will denote $\alpha \star_\xi \beta$ to converge (this will be the case, at least, if (X, ω) is monotone).

According to a terminology suggested by Manin, we will call the former the *small* quantum product, and the latter the *global* quantum product.

Remark. As we shall see in Section 2.4, these two appearently different specialisations are deeply related.

2.3. The quantum cohomology Higgs pair

In this section, we will assume that (X, ω) is monotone or more generally that the series

$$\sum_A \sum_l \frac{1}{l!} \Psi_{3,l}^A (a \otimes b \otimes c | x \otimes \cdots \otimes x)$$

converges for all a, b and c in $H_*(X)$, defining the global product $\alpha \star_\xi \beta$ by

$$\langle \alpha \star_\xi \beta, c \rangle = \sum_A \sum_l \frac{1}{l!} \Psi_{3,l}^A (a \otimes b \otimes c | x \otimes \cdots \otimes x).$$

To avoid sign problems, we will consider only the even-dimensional part of the cohomology of X. Define

$$M = \bigoplus_i H^{2i}(X; \mathbb{C}).$$

The product $\star_\xi$ defines a 1-form Ω and a Higgs pair (TM, Ω) over the manifold M. Notice that, M being a vector space, its tangent bundle has a canonical trivialisation. It is well known that:

Proposition 2.6. $d\Omega = 0$.

Proof. One just computes $(d\Omega)_\xi$ on two vector fields. As M is a vector space, one can use constant vector fields, so that the formula for the exterior derivative is simply

$$(d\Omega)_\xi (\alpha, \beta) = \alpha \cdot (\Omega_\xi(\beta)) - \beta \cdot (\Omega_\xi(\alpha)).$$

Now (using the dot $\cdot$ with two different meanings, both for differentiation along a vector field and for action of an endomorphism on a vector),

$$\langle \alpha \cdot (\Omega_\xi(\beta) \cdot \gamma), y \rangle$$
$$= \alpha \cdot (\langle \Omega_\xi(\beta) \cdot \gamma, y \rangle)$$
$$= \alpha \cdot \langle \beta \star_\xi \gamma, y \rangle$$
$$= \alpha \cdot \left(\sum_A \sum_{l \geq 0} \frac{1}{l} \Psi_{3,l}^A (b \otimes c \otimes y | x \otimes \cdots \otimes x) \right)$$
$$= \sum_A \sum_{l \geq 1} \frac{1}{(l-1)!} \Psi_{3,l}^A (b \otimes c \otimes y | a \otimes x \otimes \cdots \otimes x),$$

which is symmetric in a and b as we have noticed it in Corollary 2.3.   □

Remark. Except for the fact that we have not specified the Euler vector field, Propositions 2.5 and 2.6 together with an assumption allowing us to specialise $\star_{\xi,q}$ at $q = 1$ say that the quantum product gives the even part of the cohomology of a symplectic manifold a Frobenius structure.

The following consequence of Proposition 2.6 is mentioned in [9]:

Corollary 2.7. *Assume ξ_0 is a semi-simple point of $H^{2i}(X; \mathbb{C})$. On a neighbourhood of ξ_0, the spectral cover of the quantum product is a Lagrangian subvariety.*

Example (The case of $\mathbb{P}^1$). Let $p \in H^2(\mathbb{P}^1)$ be the generator. One easily checks that

$$p \star_{t_0 + t_1 p} p = e^{t_1}.$$

2.4. Restriction to degree-2 classes

Quantum multiplication at a given point vs. formal series. As we have noticed it above (see Section 2.2), if ξ is degree-2 class,

$$\langle \alpha \star_\xi \beta, c \rangle = \sum_A \Psi_{3,0}^A (a \otimes b \otimes c) \exp\langle \xi, A \rangle,$$

an expression which depends only on the class $[\xi]$ of ξ in $H^2(X; \mathbb{C}/2i\pi\mathbb{Z})$. Recall from Section 2.2 that the constant quantum product is given by

$$\langle \alpha \star \beta, c \rangle = \sum_A \Psi_{3,0}^A (a \otimes b \otimes c) q^A,$$

which is exactly the same thing (write $q^A = \exp\langle \xi, A \rangle$). In other words, to compute $\alpha \star_\xi \beta$ in $H^\star(X; \mathbb{C})$ amounts to the same as to compute $\alpha \star \beta$ in $H^\star(X) \otimes \Lambda$ and to specialise it at the corresponding value of q. This is a very good motivation to look at degree-2 classes.

Remark. One can define in the same mood a variant of the global quantum product: in the formula for $\star_{\xi.q}$, replace q^A by $\exp\langle \omega, A \rangle$ for some fixed class ω, usually the cohomology class of the symplectic form.

Symplectic reduction. Let us use the same notation as in Section 1.1. Consider the submanifold

$$B = H^2(X; \mathbb{C}) \xrightarrow{j} M = \oplus H^{2k}(X; \mathbb{C})$$

and apply the reduction process explained in Section 1.1 to get a Lagrangian subvariety $L_B \subset T^\star B$.

Specifically, to look at $j^\star T^\star M$ is the same thing as to look at the product $\star_\xi$ for $\xi \in H^2$. i.e., the constant quantum product. The intersection $L \cap j^\star T^\star M$ thus describes the relations in the quantum cohomology ring $QH^\star(X)$.

Example. The quantum cohomology of the complex Grassmannians has been computed e.g. in [3,21,24]. Recall that

$$QH^\star(G_k(\mathbb{C}^n)) \cong \mathbb{C}[x_1, \ldots, x_k, y_1, \ldots, y_{n-k}, q]/\mathcal{J}$$

where

- x_i is the ith Chern class of the tautological k-plane bundle,
- y_i is the ith Chern class of the orthogonal $(n-k)$-plane bundle (both x_i and y_i have degree $2i$),
- as dim $B = \dim H^2(G_n(\mathbb{C}^k); \mathbb{Z}) = 1$, one adds a single variable q; as $c_1(TG_k(\mathbb{C}^n)) = \pm n$, q has degree n; this is the multiplicative version of $\xi \in B$, more precisely, if L is the generator of $H_2(G_k(\mathbb{C}^n); \mathbb{Z})$, $q = \exp\langle \xi, L \rangle$.
- the ideal $\mathcal{J}$ is generated by the relations obtained from

$$(1 + tx_1 + \cdots + t^k x_k)(1 + ty_1 + \cdots + t^{n-k}y_{n-k}) = 1 + (-1)^{n-k}t^n q$$

by identifying the coefficients of the powers of t.

Notice moreover that $x_1, \ldots, x_k$ generate the ring $H^*(X)$, so that the ideal $\mathcal{J}$ actually describes $L \cap j^*T^*M$ in j^*T^*M. To get the reduced Lagrangian L_B, one needs just to project on the (x_1, q)-space.

To illustrate the process, here is the complete calculation in the case $k = 2, n = 4$. The coefficients of t and t^2 in the equation above give y_1 and y_2 in terms of x_1 and x_2:

$$y_1 = -x_1, \qquad y_2 = -x_2 + x_1^2.$$

The coefficients of t^3 and t^4 are then, respectively,

$$x_1^3 - 2x_1 x_2, \qquad x_1^2 x_2 - x_2^2$$

so that $QH^*(G_2(\mathbb{C}^4)) \cong \mathbb{Z}[x_1, x_2, q]/(x_1^3 - 2x_1 x_2, x_1^2 x_2 - x_2^2 - q)$. The Lagrangian L_B is

$$
\begin{aligned}
L_B &= \{(x_1, q) | \exists x_2 \text{ such that } (x_1, x_2, q) \in \mathcal{J}\} \\
&= \{(x_1, q) | x_1^4 = 2q\}.
\end{aligned}
$$

As this example shows it, the reduced Lagrangian L_B does not seem to be very meaningful in general. Let us make now a crucial assumption on the manifold X. We assume that the (classical) cohomology algebra $H^*(X)$ is generated by $H^2(X)$, its degree-2 part. Denoting by $S^*(X) = S[H^2(X; \mathbb{C})]$ the symmetric algebra on $H^2(X; \mathbb{C})$, this means that the natural ring morphism

$$S^*(X) \longrightarrow H^*(X)$$

is onto. This is the case, e.g. for projective spaces and more generally toric manifolds, complete flag manifolds, etc.

Using the notation of [10], call $(p_1, \ldots, p_k)$ a basis of $H^2(X)$. The quantum cohomology ring $QH^*(X) = (H^*(X; \mathbb{C}) \otimes \Lambda, \star)$ then consists of polynomials in p and q, with some relations. In other words, we have a surjective homomorphism:

$$S^*(H^2(X; \mathbb{C})) \otimes \mathbb{C}[H_2(X; \mathbb{Z})] \longrightarrow QH^*(X),$$

the kernel of which will be denoted by $\mathcal{J}$. The ring in the LHS is nothing other than the ring of regular functions on the cotangent bundle $T^{\star}\mathcal{B}$ of the torus $\mathcal{B} = H^2(X; \mathbb{C}/2i\pi\mathbb{Z})$. Notice that, with this remarkable notation, the symplectic form on $T^{\star}\mathcal{B}$ is $\sum \mathrm{d}p_i \wedge \mathrm{d}q_i/q_i$.

As the reduction mod $2i\pi\mathbb{Z}$ is a covering map $B \to \mathcal{B}$ we get:

Corollary 2.8 (Givental and Kim [10]). *Assume (X, ω) is a (monotone) symplectic manifold whose cohomology ring is generated by degree-2 classes. Assume that, for some value of q, the quantum product gives $H^{\star}(X)$ the structure of a semi-simple ring. Then $QH^{\star}(X)$ is the ring of functions on a Lagrangian subvariety of the cotangent bundle $T^{\star}\mathcal{B}$ of the torus $\mathcal{B} = H^2(X; \mathbb{C}/2i\pi\mathbb{Z})$.*

Example (The complex projective space). It is well-known that

$$QH^{\star}(\mathbb{P}^n) \cong \mathbb{C}[p, q, q^{-1}]/(p^{n+1} - q)$$

so that the Lagrangian is the curve $p^{n+1} = q$ in $\mathbb{C} \times \mathbb{C}^{\star}$.

Example (The plane blown-up at a point). Recall from [1] that

$$QH^{\star}(\tilde{\mathbb{P}}^2) \cong \mathbb{C}[p_1, p_2, q_1^{\pm 1} q_2^{\pm 1}]/(p_1^2 + p_2^2 - p_2 q_2 - 2q_1 q_2^{-1}, p_1 p_2 + q_1 q_2^{-1}).$$

This is an easy exercise [5] to check that these relations indeed define a Lagrangian in $T^{\star}(\mathbb{C}^{\star})^2$.

Let us conclude by a list of remarks on this result.

The semi-simplicity assumption. According to a conjecture of Tian [23], the semi-simplicity assumption might be automatically satisfied in Fano manifolds. There are (non-monotone) Kähler manifolds for which this is not satisfied. The most obvious example is that of $K3$ surfaces, whose quantum cohomology ring is nilpotent.

Poisson commuting relations. This result (Corollary 2.8) was proved by Givental and Kim as a comment of their theorem on flag manifolds. Recall that they have found that, in this case, the Lagrangian subvariety was a common level set of first integrals of the Toda lattice. Notice that Corollary 2.8 says that the ideal $\mathcal{J}$ is stable under Poisson bracket:

$$f, g \in \mathcal{J} \Rightarrow \{f, g\} \in \mathcal{J}.$$

In the case of flag manifolds, the property they get is much stronger: there exists a system $(f_1, \ldots, f_k)$ of generators of $\mathcal{J}$ which Poisson commute.

This is not the case in general: one can check for instance that the ideal defining the quantum cohomology of the plane blown-up at a point cannot be generated by two Poissson commuting elements. Moreover, this property of being defined by an integrable

---

[5] In the computation for this example in [17], a sign is wrong. This was also the case in the preliminary version of [1]. I noticed the mistake exactly because I was trying to check Corollory 2.8 on this example.

system would imply that L_B is a complete intersection and it is quite easy to find examples in which L_B is not, e.g. $\mathbb{P}^2$ blown-up at two points (this is left as an exercise for the reader).

Homogeneity property and the first Chern class. Recall now that $QH^*(X)$ is a graded ring, the grading being defined by the natural grading on $H^*(X)$ and by the first Chern class on Λ: if $A \in H_2(X; \mathbb{Z})$, $\deg q^A = 2\langle c_1, A \rangle$. As a consequence, the ideal $\mathcal{J}$ is quasi-homogeneous. Let η be the vector field on $\mathcal{B}$ which is the generator of the $\mathbb{C}^*$-action

$$e^z \cdot ([\xi], a) = ([\xi + zc_1], e^z a).$$

Write the Cartan formula $\mathcal{L}_\eta = di_\eta + i_\eta d$ and apply it to the Liouville form. Obviously $\mathcal{L}_\eta \lambda = \lambda$ and $i_\eta \lambda = c_1$, so that, if $j : L \subset T^*\mathcal{B}$ is the inclusion,

$$j^*\lambda = \mathcal{L}_\eta(j^*\lambda) = di_\eta j^*\lambda + i_\eta dj^*\lambda.$$

As $j^*\lambda$ is closed, we get

$$j^*\lambda = d(c_1|_L).$$

Thus, the first Chern class is a primitive of the Liouville form on L.

This is of course the "degree-2" version of the homogeneity property with respect to the Euler vector field mentioned in Corollary 1.9.

Closedness of the form Ω and degree-2 classes. To prove that $d\Omega = 0$ (Proposition 2.6), we have used the symmetry property of Gromov–Witten invariants enclosed in Proposition 2.2. If one is interested only in degree-2 classes, it can be noticed that this is a consequence of the property (a variant of Proposition 2.4):

Proposition 2.9. *Let (X, ω) be a weakly monotone symplectic manifold of dimension $2n$. Let A be a non-zero element of $H_2(X; \mathbb{Z})$. For any $a, b \in H_*(X; \mathbb{Z})$,*

$$\Psi^A_{3,0}(a \otimes b \otimes x) = \langle \xi, A \rangle \Psi^A_{2,0}(a \otimes b).$$

The proof of Proposition 2.6 can be replaced by:

$$\langle \alpha \cdot (\Omega_\xi(\beta) \cdot \gamma), y \rangle$$
$$= \alpha \cdot (\langle \Omega_\xi(\beta) \cdot \gamma, y \rangle)$$
$$= \alpha \cdot \left(\sum_A \Psi^A_{3,0}(b \otimes c \otimes y) \exp\langle , \xi, A \rangle \right)$$
$$= \sum_{A \neq 0} \Psi^A_{3,0}(b \otimes c \otimes y)\langle \alpha, A \rangle \exp\langle , \xi, A \rangle$$
$$\quad \text{(as the derivative of the } A = 0\text{-term vanishes)}$$
$$= \sum_{A \neq 0} \Psi^A_{2,0}(c \otimes y)\langle \beta, A \rangle \langle \alpha, A \rangle \exp\langle \xi, A \rangle$$

(as $\deg \beta = 2$), an expression which is symmetric in α and β.

It is also easy to find a primitive of Ω. Precisely, let

$$S : H^2(X; \mathbb{C}) \longrightarrow \mathrm{End}(H^*(X; \mathbb{C}))$$

be the mapping defined by

$$\langle S(\xi) \cdot \alpha, b \rangle = (a \cdot b \cdot x) + \sum_{A \neq 0} \Psi_{2,0}^A (a \otimes b) \exp\langle \xi, A \rangle.$$

It is easily checked that $\mathrm{d}S = \Omega$.

Acknowledgements

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